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2003 J. Phys. A: Math. Gen. 36 99

(http://iopscience.iop.org/0305-4470/36/1/307)

J. Phys. A: Math. Gen. **36** (2003) 99–121 PII: S0305-4470(03)52835-9

Fundamental cycle of a periodic box–ball system

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Received 28 August 2002, in final form 30 October 2002
Published 10 December 2002
Online at stacks.iop.org/JPhysA/36/99

Abstract
We investigate a box–ball system with periodic boundary conditions. Since the box–ball system is a deterministic dynamical system that takes only a finite number of states, it will exhibit periodic motion. We determine its fundamental cycle for a given initial state.

PACS numbers: 05.65.+b, 02.30.Ik, 05.45.Yv

1. Preface

A cellular automaton (CA) is a discrete dynamical system consisting of a regular array of cells [1]. Each cell takes only a finite number of states and is updated in discrete time steps. Although the updating rules are simple, CAs often exhibit very complicated time evolution patterns which resemble natural phenomena such as chemical reactions, turbulent flow, nonlinear dispersive waves and solitons. A typical CA exhibiting a solitonic behaviour is the box and ball system (BBS) which is a reinterpretation of the dynamical system of '10'-sequence proposed by Takahashi and Satsuma [2, 3]. (Precisely speaking, the BBS is not a CA for its time evolution rule is nonlocal and might be called filter automaton or filter cellular automaton.) The BBS is *integrable* in the sense that it is obtained from the KdV equation through a limiting procedure called ultradiscretization [4]. It can also be obtained from a two-dimensional integrable lattice model and its relation to combinatorial R matrices of $U'_q\left(A_N^{(1)}\right)$ is well established [5, 6].

The original box–ball system is defined as a dynamical system of a finite number of balls in an *infinite* number of one-dimensional array of boxes. However, it is possible to extend the time evolution rule to a system consisting of a *finite* number of boxes with periodic boundary conditions [7]. The BBS with periodic boundary condition (pBBS) is also connected to the combinatorial R matrix of $U'_q\left(A_N^{(1)}\right)$, and its time evolution rule is represented as a Boolean recurrence formula related to the algorithm for calculating the $2N$th root. Since the pBBS is composed of a finite number of cells, and it can only take on a finite number of patterns, the time evolution of the pBBS is necessarily periodic. In the present paper, we investigate the

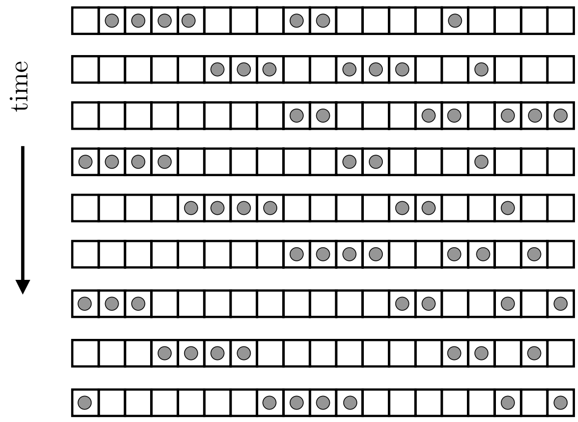

Figure 1. Time evolution of a pBBS.

fundamental cycle, i.e., the shortest period of the discrete periodic motion of the pBBS.[3] In section 2, we review the pBBS and present its conserved quantities. The formula for the total number of patterns for a given set of conserved quantities is also presented. In section 3, we define several notions which are necessary to prove the formula for the fundamental cycle. The fundamental cycle for an arbitrary initial state is derived in section 4. Section 5 is devoted to the concluding remarks.

2. Periodic box and ball system (pBBS)

Let us consider a one-dimensional array of N boxes. To impose a periodic boundary condition, we assume that the Nth box is the adjacent box to the first one. (We may imagine that the boxes are arranged in a circle.) The box capacity is 1 for all the boxes, and each box is either empty or filled with a ball at any time step. We denote the number of balls by M, such that $M < \frac{N}{2}$. The balls are moved according to a deterministic time evolution rule. There are several equivalent ways to describe this rule. For example,

1. In each filled box, create a copy of the ball.
2. Move all the copies once according to the following rules.
3. Choose one of the copies and move it to the nearest empty box on the right of it.
4. Choose one of the remaining copies and move it to the nearest empty box on the right of it.
5. Repeat the above procedure until all the copies have been moved.
6. Delete all the original balls.

An example of the time evolution of the pBBS according to this rule is shown in figure 1. It is not difficult to prove that the result obtained does not depend on the choice of the copies at each stage and that it coincides with the evolution rule of the original BBS when N goes to infinity. An advantage of the above description is that we can easily extend the evolution rule to the system with many kinds of balls and many kinds of box capacities. In the present paper, however, we restrict ourselves to the pBBS with only one kind of ball and with box capacity 1, and we revert to the following description which is more convenient to determine the fundamental cycle. We denote an empty box by '0' and a filled box by '1'. Then the pBBS

[3] Part of the present work was already announced in [8].

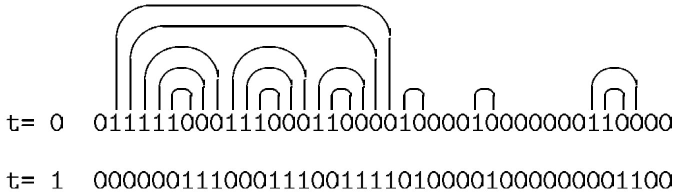

Figure 2. Time evolution rule for a pBBS expressed as 10 sequences.

t= 0	000011110001100	00001110010	0000110
t= 1	100000001110011	10000001101	1000001
t= 2	011110000001100	01110000010	0110000
t= 3	000001111000011	00001110001	0001100
t= 4	110000000111100	10000001110	0000011
t= 5	001111000000011	01110000001	1100000
t= 6	110000111100000	10001110000	0011000
t= 7	001100000011110	01000001110	0000110
	(4,2)	(3,1)	(2,0)

Figure 3. Examples of time evolution patterns of pBBSs. The middle (right) pBBS is obtained from the left (middle) pBBS by 10-elimination. The numbers in the parentheses denote lengths of *solitons* in the patterns.

is regarded as a dynamical system of finite sequence of '0's and '1's. Because of the periodic boundary condition, we consider the last entry in the sequence to be adjacent to the first entry. (In this sense, the sequence should be regarded as a ring which consists of '0's and '1's.) We say two sequences are *equivalent* if one coincides with the other by translation. For example, the sequence 011101100100001 is equivalent to 101100100001011. The updating rule of the sequence is given as

1. Connect all the '10' pairs in the sequence with arc lines.
2. Neglecting the 10 pairs which were connected in the first step, connect all the remaining '10's with arc lines.
3. Repeat the above procedure until all the '1's are connected to '0's.
4. Interchange every '1' and '0' which are connected to each other by an arcline.

This rule is illustrated in figure 2 and some examples of time evolutions of the pBBS are shown in figure 3.

Just as the original BBS [9], the pBBS has a number of conserved quantities. Let p_1 be the number of 10 pairs in a state of the pBBS, which are connected with arc lines in the first step of the above time evolution rule. Similarly, we denote by p_2 the number of 10 pairs connected in the second step, by p_3 those in the third step, ... and by p_l those in the last step. Clearly $p_1 \geqslant p_2 \geqslant \cdots \geqslant p_l$. For example, in the state shown in figure 2, $p_1 = 6$, $p_2 = 4$, $p_3 = 2$, $p_4 = 1$, $p_5 = 1$.

Proposition 2.1. *The numbers p_i in the weakly decreasing series of integer $\{p_1, p_2, \ldots, p_l\}$ are conserved quantities for the time evolution of the pBBS.*

To prove proposition 2.1, we prepare two lemmas. The number of '10's in a state of the pBBS is equal to that of '01's, for they coincide with the number of the set of consecutive '1's (or '0's). According to the evolution rule, a '10' at t turns into a '01' at $t + 1$ and all the '01's at $t+1$ are created in this way. Hence we find that the number of '10's at t is equal to that of '01's at $t+1$. Thus we have the following lemma.

Lemma 2.1. *The number of* '10'*s in the pBBS does not change in time evolution. It also coincides with the number of* '01'*s at each time step.*

When we eliminate the '10's in a state, we obtain a new sequence. Hereafter we call this operation 10-*elimination*. (In order to associate this new sequence with the state of a pBBS with a smaller number of boxes and balls, we have to determine the position of the first entry of the sequence. We shall consider this problem in the subsequent section, however, as it is not necessary for the proof considered here.) The operation 01-*elimination* is defined in a similar way. By 10-elimination, all the series of consecutive '1's and consecutive '0's in the state have their length reduced by one. The 01-elimination has the same effect on the state, hence, the sequence obtained by 01-elimination is equivalent to that obtained by 10-elimination. When the sequence obtained by 10-elimination is updated according to the time evolution rule, it becomes equivalent to that obtained by 01-elimination on the state in the next time step. (This fact is easily understood from figure 2.) Thus we have

Lemma 2.2. *Both 10-elimination and 01-elimination commute with the updating rule of the pBBS, that is, the sequence obtained by 10-elimination (or 01-elimination) at* $t+1$ *is equivalent to the one updated from the sequence obtained by 10-elimination (or 01-elimination) at t.*

Proof of proposition 2.1. From lemma 2.1, it is found that p_1 does not change in time. Since p_2 is the number of '10's in the sequence obtained by 10-elimination, from lemmas 2.1 and 2.2, it is also a conserved quantity in time. Repeating the 10-elimination and using lemmas 2.1 and 2.2, we find that all the p_j are conserved in time. □

Since $\{p_1, p_2, \ldots, p_l\}$ is a weakly decreasing series of positive integers, we can associate it with a Young diagram with p_j boxes in the jth column ($j = 1, 2, \ldots, l$). Then the lengths of the rows are also weakly decreasing positive integers, and we denote them

$$\{\underbrace{L_1, L_1, \ldots, L_1}_{n_1}, \underbrace{L_2, L_2, \ldots, L_2}_{n_2}, \ldots, \underbrace{L_s, L_s, \ldots, L_s}_{n_s}\}$$

where $L_1 > L_2 > \cdots > L_s$. The set $\{L_j, n_j\}_{j=1}^{s}$ is an alternative expression of the conserved quantities of the system. Let $\ell_0 := N - \sum_{j=1}^{l} 2p_j = N - \sum_{j=1}^{s} 2n_j L_j$, and

$$\ell_j := L_j - L_{j+1} \qquad (j = 1, 2, \ldots, s-1) \tag{2.1}$$

$$N_j := \ell_0 + 2n_1(L_1 - L_{j+1}) + 2n_2(L_2 - L_{j+1}) + \cdots + 2n_j(L_j - L_{j+1}). \tag{2.2}$$

Then, for a fixed number of boxes N and conserved quantities $\{L_j, n_j\}$, the number of possible states of the pBBS $\Omega(N; \{L_j, n_j\})$ is given by the following proposition.

Proposition 2.2

$$\Omega(N; \{L_j, n_j\}) = \frac{N}{\ell_0}\binom{\ell_0 + n_1 - 1}{n_1}\binom{N_1 + n_2 - 1}{n_2}\binom{N_2 + n_3 - 1}{n_3} \times \cdots \times \binom{N_{s-1} + n_s - 1}{n_s}. \tag{2.3}$$

Although the proposition can be proved by elementary combinatorial arguments, it is convenient to use notions defined in subsequent sections and we thus refer the proof to appendix A. The fact, however, that the right-hand side of (2.3) is an integer is confirmed using the following lemma.

Lemma 2.3. *Let $\{a_i\}_{i=1}^s$, $\{b_i\}_{i=1}^s$ be 2s positive integers and suppose that the $\{a_i\}_{i=1}^s$ do not have a common divisor, i.e., $GCD(a_1, a_2, \ldots, a_s) = 1$. If there exists a k such that*

$$\frac{b_1}{a_1} = \frac{b_2}{a_2} = \cdots = \frac{b_s}{a_s} = k$$

then k is an integer.

3. Soliton in pBBS and its properties

Let the boxes be numbered $1, 2, 3, \ldots, N$ from right to left. Accordingly, the position of an entry in the associated 01 sequence is denoted by the number of the corresponding box. Because of the periodic boundary condition, we always use the convention that the numbers are defined in $\mathbb{Z}_N$, that is, $j \equiv j + N$, and an equality such as $i < j < k$ means that $j \in \{i + 1, i + 2, \ldots, k - 1\}$. In order to explain some important features of the pBBS, we assign an integer index to every entry of the sequences obtained at each time step in the updating rule. Precisely speaking, we define the map $\phi_t : \mathbb{Z}_N \to \mathbb{Z} \sqcup \{-\infty\}$ in the following way.

1. Choose the 10 pairs in the construction of the conserved quantity p_1. If the positions of '1's and '0's are $i_1, i_2, \ldots, i_{p_1}$ and $j_1, j_2, \ldots, j_{p_1}$ respectively, then $\phi_t(i_k) = 1, \phi_t(j_k) = -1$ $(1 \leqslant k \leqslant p_1)$.
2. Next, choose the 10 pairs for the conserved quantity p_2. If the positions of '1's and '0's are $i_1, i_2, \ldots, i_{p_2}$ and $j_1, j_2, \ldots, j_{p_2}$ respectively, then $\phi_t(i_k) = 2, \phi_t(j_k) = -2$ $(1 \leqslant k \leqslant p_2)$.
3. Similarly, the indices $3, -3, 4, -4, \ldots$ are assigned to the positions of the 10 pairs in the construction of $p_3, p_4, \ldots$.
4. For the positions of '0's which are not connected with '1's in the construction of the conserved quantities, the indices are $-\infty$.

We say that the entry ('0' or '1') in the position j at time t has index $\phi_t(j)$. From this definition, we note that n_1 entries take the largest index L_1. Furthermore, if we denote the number of entries of index k by I_k, then I_k does not change in time and is given as

$$I_k = \begin{cases} \ell_0 & \text{for} \quad k = -\infty \\ n_1 & \text{for} \quad L_2 + 1 \leqslant |k| \leqslant L_1 \\ n_1 + n_2 & \text{for} \quad L_3 + 1 \leqslant |k| \leqslant L_2. \\ \cdots & \\ \sum_{j=1}^s n_j & \text{for} \quad 1 \leqslant |k| \leqslant L_s \end{cases} \tag{3.1}$$

For example, the indices of the sequence

0 0 0 1 1 1 0 0 1 1 0 1 1 0 0 0

are given as

-4 $-\infty$ $-\infty$ 4 2 1 -1 -2 3 1 -1 2 1 -1 -2 -3

Using the index, we define *solitons* and their position.

Definition 3.1. *A soliton consists of* '1'*s in a 01 sequence and can be determined by the following process.*

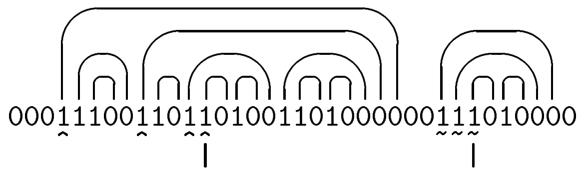

Figure 4. Solitons in a pBBS.

1. *Choose one of the* '1'*s whose index is* L_1. *We suppose its position is* j_{L_1}.
2. *Choose the* '1' *which is the nearest* '1' *with index* $L_1 - 1$ *in the anticlockwise direction from the position* j_{L_1} *(i.e. to the 'right' of that position.) Let* j_{L_1-1} *be its position.*
3. *Similarly, let* j_{L_1-2} *be the position of its (anticlockwise) nearest* '1' *with index* $L_1 - 2$.
4. *Repeat the procedure until the* '1' *at* j_1 *is determined.*
5. *Then we define the set of* '1'*s at* $j_{L_1}, j_{L_1-1}, \ldots, j_1$ *to constitute a soliton with length* L_1 *and position* j_1.
6. *Perform the same procedure starting from one of the other* $n_1 - 1$ '1''*s with index* L_1*, and repeat it until all the* n_1 *solitons with length* L_1 *have been determined.*
7. *The largest index of the remaining* '1'*s is* L_2. *Repeat the above procedure to the remaining* '1'*s and determine all* n_2 *solitons with length* L_2.
8. *In a similar manner, we determine* n_3 *solitons with length* L_3*,* n_4 *solitons with length* $L_4, \ldots$ *and finally we determine* n_s *solitons with length* L_s.

In figure 4, there are eight solitons (length $4 \cdots 1$, length $3 \cdots 1$, length $2 \cdots 2$ and length $1 \cdots 4$). The '1's with '$\wedge$' at the bottom constitute the largest soliton and those with '$\sim$' constitute the second largest soliton. Their position is marked by '|'.

Next, we define the notion of *a block*. Recall that we drew arc lines between 10 pairs to determine the time evolution of pBBS. As shown in figure 4, a state of the pBBS is divided into disjoint 01 sequences of such arc lines. We call each disjoint sequence *a block*. For example, there are two blocks in figure 4. The following properties of a block are obvious from its definition.

1. The numbers of '0's and '1's in a block are the same.
2. There is only one soliton with the largest length in each block.
3. After time evolution $t \to t + 1$, all the '0's in a block at t turn into '1's, and all the '1's into '0's.
4. If the length of the largest soliton in a block is L_M, the '1' at the left edge of the block has index L_M and the '0' at the right edge of the block has index $-L_M$.
5. The length of a block is twice the sum of the lengths of solitons contained in the block.

The following proposition, which plays an important role in subsequent proofs, is a direct consequence of the definition of soliton and block.

Proposition 3.1. *Suppose that* '1'*s at the positions* $j_L, j_{L-1}, \ldots, j_1$ *constitute a soliton with length L, and that the* '0'*s which form the 10 pairs with these* '1'*s are at the positions* $i_1, i_2, \ldots, i_L$. *The* '1' *at* j_k *has index k and the* '0' *at* i_k *has index* $-k$. *Then,*

1. *If an entry is located between* j_{k+1} *and* j_k*, the absolute value of its index is less than k.*
2. *If an entry is located between* i_k *and* i_{k+1}*, the absolute value of its index is less than or equal to k.*

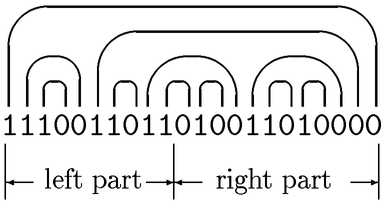

Figure 5. An example of a block.

3. *If we eliminate all the* '1'*s at* $j_L, j_{L-1}, \ldots, j_1$ *and all* '0'*s at* $i_1, i_2, \ldots, i_L$*, the entries with positions between* j_{k+1} *and* j_k *and* i_k *and* $i_{k+1} (k = 1, 2, \ldots, L)$ *constitute disjoint blocks.*
4. *Let* $\Delta(k)$ *be the difference of the number of* '1'*s and* '0'*s contained between* j_L *and k, then* $0 \leqslant \Delta(k) \leqslant L$*. In particular,* $\Delta(k) = 0$ *iff* $k = i_L$ *and* $\Delta(K) \leqslant L - 1$ *for* $k < j_1$*. Similarly, let* $\Delta'(k)$ *be the difference of the number of* '0'*s and* '1'*s contained between k and* i_L*, then* $0 \leqslant \Delta'(k) \leqslant L$*. In particular,* $\Delta'(k) = 0$ *iff* $k = j_L$*.*

Let us discuss some important properties of such a block. We assume that the length of the largest soliton in the block is *L*, and that it is constituted by '1's at positions $j_L, j_{L-1}, \ldots, j_1$, where '1' at j_k forms a 10 pair with '0' at i_k $(1 \leqslant k \leqslant L)$. First we divide a block into two parts (figure 5). The *right part* of a block is the 01 sequence which is located on the right-hand side (anticlockwise) from the position of the largest soliton in the block. In other words, the right part is the sequence from i_1 to i_L. The remainder is called the *left part* of the block. It is the sequence from j_L to j_1. The following properties are obvious.

1. The number of '1's in the left part is greater by *L* than that of '0's. In contrast, the number of '1's in the right part is less by *L* than that of '0's.
2. The entries at the edges of the left part are '1' and those of the right part are '0'.

Furthermore, from the rule for making 10 pairs, we have the following proposition.

Proposition 3.2

1. *In any sequence starting from the left edge of the left part, the number of* '1'*s is greater than that of* '0'*s.*
2. *In any sequence ending at the right edge of the left part, the number of* '1'*s is greater than that of* '0'*s.*
3. *In any sequence ending at the right edge of the right part, the number of* '0'*s is greater than that of* '1'*s.*
4. *In any sequence starting from the left edge of the right part, the number of* '0'*s is greater than or equal to that of* '1'*s.*

Proof. Statements 1 and 3 are obvious. For statement 2, we assume that the sequence starts from position *k* and that $j_{m+1} \leqslant k \leqslant j_m$. From proposition 3.1, there can only be solitons with length less than or equal to $m - 1$ between j_{m+1} and j_m. From proposition 3.1 (statements 2 and 3), the difference of the number of '0's and '1's between *k* and $j_m - 1$ is less than or equal to $m - 1$. On the other hand, there are *m* more '1's than '0's between j_m and j_1. Hence statement 2 holds. Statement 4 is proved in a similar manner. □

Next we discuss some features of the time evolution of solitons. A state of pBBS at time step $t - 1$ is divided into blocks. We pay attention to a single one. The following proposition is essential to understand the movement of solitons.

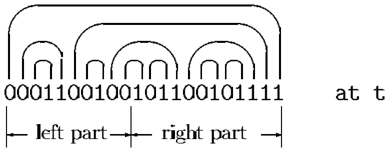

Figure 6. The left and right parts at t updated from the block at $t-1$ shown in figure 5.

Proposition 3.3. *In a block, the number and type of solitons at* $t-1$ *do not change at t. Furthermore, the position of the solitons at t does not depend on the pattern outside the block at* $t-1$.

The above proposition is proved by induction using the lemmas given below. Let L be the length of the largest soliton in the block. The position of its components is denoted by $j_L, j_{L-1}, \ldots, j_1$ and that of the pairing '0's by $i_1, i_2, \ldots, i_L$. We refer to the sequence which, at t, is updated from the left (right) part of the block at $t-1$ as the left (right) part at t. Figure 6 shows the left and right parts at t which are updated from the state shown in figure 5.

Lemma 3.1. *For* $L \geqslant 3$, *if* '10'*s are eliminated in the left part at t, there are at least three consecutive* '0'*s in the left edge. If further 10-elimination is possible and is performed on the sequence, there are at least four consecutive* '0'*s in the left edge. In general, if 10-elimination is performed m times* $(1 \leqslant m \leqslant L-2)$, *there remain at least* $m+2$ *consecutive* '0'*s in the left edge.*

Proof. Recall that all the '0's in a block at $t-1$ turn into '1's, and all the '1's into '0's. From proposition 3.2 (statement 2) we have that every '1' in the left part at t has its pair '0' within the left part. Let $\Delta(k)$ be the difference of the numbers of '0's and '1's between j_L and k. From proposition 3.2 (statement 1) or 3.1 (statement 4) we find that $\min_{k\in[j_L,j_1]} \Delta(k) \geqslant 1$. Suppose that $\Delta_k = 1$ at $k = k_i$ $(1 \leqslant i \leqslant r)$, then the entry at k_i is '1' and the entry at $k_i + 1$ is '0'. Hence, if we eliminate '10's, the '1's at k_i are removed. Let J_2 be the set of positions $J_2 = \{k | k \neq k_i, k_i + 1\ (1 \leqslant i \leqslant r), k \in [j_L + 1, j_1]\}$. Since $\min_{k\in J_2} \Delta(k) \geqslant 2$, there are at least three '0's in the left edge of the reduced sequence. To prove the lemma, proceed in a similar manner. □

Lemma 3.2. *Suppose that* $L \geqslant 3$. *In the left part at t, if we perform 10-elimination for* $L-2$ *times, only L consecutive* '0'*s remain and no* '1' *is left.*

Proof. As in the proof of lemma 3.1, we consider the difference $\Delta(k)$. Since 10-elimination always erases a '1' and '0' simultaneously, the value $\Delta(k)$ also indicates the difference of '0' and '1' in the reduced sequence, between its left edge j_L and the entry at the (original) position k. From proposition 3.1 (statement 4), $\Delta(k)$ $(j_L \leqslant k \leqslant j_1)$ can only take its maximum value L at $k = j_1$. On the other hand, from lemma 3.1, at least L consecutive '0's remain after $L-2$ 10-eliminations. Hence only L consecutive '0's can remain in the left part at t. □

Lemma 3.3. *In the left part, the number and type of solitons at* $t-1$ *do not change at t except for the largest soliton. Furthermore, the position of the solitons at t does not depend on the pattern outside the block at* $t-1$.

Proof. If $L \leqslant 2$, there is no '1' in the left part at t and the above statement becomes trivial. We therefore suppose $L \geqslant 3$. Consider a state of a smaller pBBS which has a left part of the same size. We assume that the pattern of the state coincides with the left part at $t-1$ except for the '1's which constitute the largest soliton, all of which are replaced with '0's. The number of ‘0’s in this state is greater than that of ‘1’s, by an amount L. If we update according to the time evolution rule, this pattern is exactly equal to that of the left part at t (proposition 3.1 (statement 3)). Since the number and type of solitons do not change in the sequence (proposition 2.1), to prove the lemma, it suffices to show that no ‘1’ of the left part belongs to a soliton whose position is outside that part and that no ‘1’ from the outside belongs to a soliton whose position is in the left part.

From proposition 3.2 (statement 2), ‘1’ does not form a pair with ‘0’ at the right edge (at the position j_1), which, at the same time, means that no ‘1’ forms a pair with ‘0’ in the outside. (Remember that a pair means a ‘1’ and a ‘0’ which are connected by an arc line according to the time evolution rule at t.) Since the pattern of the left part at t is the same as the sequence considered above, the indices of ‘1’s coincide with those of that sequence. Hence the largest index of the left part at t is at most $L - 2$.

There are L more ‘0’s than ‘1’s in the left part at t, and the index of the ‘0’ at j_1 is less than or equal to $-L$. Since the largest index of the left part is at most $L - 2$, from proposition 3.1 (statement 1), a ‘1’ in the left part does not belong to a soliton whose position is outside the left part.

There are at least two consecutive ‘0’s in the left edge of the left part at t. The index of the second ‘0’ from the left edge is at most -2, and, by proposition 3.1 (statement 1), no ‘1’ in the outside belongs to a soliton whose position is in the left part if its index is less than or equal to -2.

Suppose that a ‘1’ of index k ($k \geqslant 3$) outside the left part constitutes a soliton with ‘1’s inside whose indices are $k - 1, k - 2, \ldots, 1$. When we perform 10-elimination $k - 2$ times, from lemma 3.1 we have that there are at least two consecutive ‘0’s between the inside ‘1’ with index $k - 1$ and the outside ‘1’ of index k, which contradicts our definition of a soliton.

Thus we find that no ‘1’ on the outside constitutes a soliton whose position was in the left part, which completes the proof. □

As for the right part, we have the following lemmas.

Lemma 3.4. *For $L \geqslant 2$, if* ‘10’*s are eliminated in the right part at t, there are at least three consecutive* ‘1’*s in the right edge. If possible, further 10-elimination is performed, there are at least four consecutive* ‘1’*s in the right edge. In general, if 10-elimination is performed m times* ($1 \leqslant m \leqslant L - 2$)*, there remain at least $m + 2$ consecutive* ‘1’*s in the right edge.*

The proof is analogous to that of lemma 3.1.

Lemma 3.5. *If we eliminate* ‘10’*s in the right part at t*, $L - 1$ *times, the reduced sequence consists of L consecutive* ‘1’*s and no* ‘0’.

Proof. Let J_m ($1 \leqslant m \leqslant L - 2$) be the set of positions of the entries in the right part at t which remain after m10-eliminations and do not constitute $m + 2$ consecutive ‘1’s in the right edge. Then, as in the proof of lemma 3.1, we have an inequality: $\min_{k \in J_m} \Delta'(k) = m + 1$. For $m = L - 2$, this inequality implies that the sequence has the form

$$\underbrace{1010\cdots10}_{\text{repetition of }10}\ \underbrace{11\cdots1}_{L}.$$

Hence, by applying 10-elimination $L - 1$ times, we obtain a pattern with L consecutive ‘1’s and no ‘0’. □

Now we give the proof of proposition 3.3

Proof of proposition 3.3. From proposition 3.2 (statement 4), any sequence starting from the left edge of the right part at t contains a number of '1's, larger than or equal to that of '0's. (Recall that '0' and '1' in a block at $t-1$ turn to '1' and '0' at t.) Hence a '0' in the right part at t forms a pair with one of the '1's in the right part, and the pair is not affected by the outside pattern. There are just L '1's which do not form pairs with inside '0's. These '1's are obtained by 10-eliminations as in lemma 3.5. If their indices at t are $L, L-1, \ldots, 2, 1$, then since the index of '0' at j_1 is at most $-L$ and all the indices in the right part are less than or equal to L (lemma 3.5), no '1' outside the right block can belong to the inside solitons (proposition 3.1 (statement 1)) and the L '1's constitute a soliton with length L whose position does not depend on the outside pattern. In fact, its position is always i_L. Then, together with lemma 3.3, the solitons in the block do not depend on the outside pattern. In other words, the positions of solitons at t exactly coincide with those in the case where no other block exists in the state. According to proposition 2.1, the solitons are preserved in time evolution and the proposition is proved. Therefore we now only have to show that the indices of the L consecutive '1's, which are obtained by 10-eliminations in the right block at t, are $L, L-1, \ldots, 2, 1$.

By the definition of a block, the entry at $i_L + 1$ is necessarily '0', and the '1' at the right edge i_L forms a pair with it. Hence its index is 1 and the statement is true for $L = 1$.

When we eliminate all the '10's in the state, the right edge of the 'reduced' block is '1' (lemma 3.4). Since all the blocks for which the largest soliton was of length 1 are eliminated, and since the other blocks have at least one '0' in the left edge, this '1' forms a pair with a '0' and its index is 2 and the statement is true for $L \leqslant 2$. Similarly, when we eliminate the '10's once more, the right edge of the reduced block is still '1' (lemma 3.4). Since all the blocks with the largest soliton whose length $L \leqslant 2$ are eliminated, lemma 3.1 assures that the next right entry of '1' is 3.

Repeating this argument, we find that the consecutive '1's have indices $L, L-1, \ldots, 1$, which completes the proof. □

In the above proof, we have also shown that the position of the largest soliton at t is the right edge of the block. From proposition 3.1 (statement 3) and the fact that the length of the block is twice the sum of the lengths of solitons in the block, we obtain the following corollary.

Corollary 3.1. *The position of the largest soliton is updated to the right edge of the block. The difference between the position of the largest soliton at t and that at $t-1$ is $L + 2\times$ (sum of the lengths of solitons in the right part).*

Since the '0's in the right part at t form pairs with the '1's in the right part, we have

Proposition 3.4. *The pattern of the right part at $t-1$ coincides with the pattern in the same region at $t+1$.*

Finally the movement of solitons is expressed by the following theorem.

Theorem 3.1. *Suppose that a pBBS has M solitons. Let $X_j(t-1)$ and L_j $(1 \leqslant j \leqslant M)$ be the positions and lengths of the solitons at $t-1$. Then, their positions at t, $X_j(t)$ $(1 \leqslant j \leqslant M)$, satisfy*

$$X_j(t) - X_j(t-1) \equiv L_j + \sum_{i=1}^{M} 2x_j^i(t)\min[L_j, L_i] \text{ modulo } N \tag{3.2}$$

where the soliton at $X_j(t)$ has length L_j and the coefficients $x_j^i(t)$ $(1 \leqslant i, j \leqslant M)$ satisfy $x_j^i(t) = -x_i^j(t) \in \{-1, 0, 1\}$. In particular, if $X_s(t)$ is the position of the largest soliton, $x_s^i(t) = 0$ or 1.

Proof. We prove the theorem by induction on the number of solitons M. For $M = 1$, the statement is trivial $\left(x_1^1(t) = 0\right)$. Let $k \geqslant 1$ and assume that the theorem is true for $M \leqslant k$. When there are $k + 1$ solitons, if the state at $t - 1$ consists of multiple blocks, the latter part of proposition 3.3 can be applied to each block and the theorem is proved. When the state consists of one block, as shown in the proof of lemma 3.3, except for the largest soliton the other solitons in the left part are updated as if the largest soliton did not exist. Hence they satisfy (3.2) by induction hypothesis. Let $X_s(t - 1)$ be the position of the largest soliton at $t - 1$. From corollary 3.1,

$$X_s(t) - X_s(t-1) = L_s + 2 \sum_{j \in \{X_j(t-1) \text{ belongs to the right part}\}} L_j$$
$$= L_s + \sum_j 2x_s^j(t) \min[L_s, L_j] \tag{3.3}$$

where $x_s^j(t) = 1$ if the jth soliton is in the right part and otherwise $x_j^s(t) = 0$. Hence the largest soliton satisfies (3.2).

On the other hand, by proposition 3.4, the solitons in the right part at t are supposed to move to their positions at $t - 1$ if there is no soliton outside the region. By the induction hypothesis, this movement satisfies (3.2). Then, if the jth soliton is in the right part at t, $^\exists x'^i_j \in \{-1, 0, 1\}$,

$$X_j(t-1) - X_j(t) \equiv L_j + \sum_{i=1, i \neq s}^{M} 2x'^i_j \min[L_j, L_i] \text{ modulo } N \tag{3.4}$$

where $x'^i_j = -x'^j_i$ and $x'^i_j = 0$ when the ith or jth soliton is outside the right part. Since $x_s^j(t) = 1$ for the jth soliton in the right part, if we put $x_j^s(t) = -1$, $x_j^i(t) = -x'^j_i\,(i, j \neq s)$, then

$$X_j(t) - X_j(t-1) \equiv L_j + \sum_{i=1}^{M} 2x_j^i(t) \min[L_j, L_i] \text{ modulo } N. \tag{3.5}$$

Thus the theorem also holds for $M = k + 1$, which completes the proof. □

So far, we have not obtained concrete expressions for $x_i^j(t)$; however, theorem 3.1 and corollary 3.1 are enough to determine the fundamental cycle of the pBBS for a given initial state.

In the subsequent section, we shall use some properties of solitons with length 1. Hereafter we often refer to such a soliton as a 1-soliton. The following proposition is easily obtained from proposition 3.1

Proposition 3.5. *Let p be the position of a 1-soliton. The entry at $p + 1$ is necessarily a* '0'. *If the entry at $p - 1$ is* '1', *then its index is greater than 2. In contrast, if the entries at $p - 1, p, p + 1$ are* '010', *then p is the position of a 1-soliton. Similarly if the entries at $p - 1, p, p + 1$ are* '110' *and the index of* '1' *at $p - 1$ is greater than 2, then p is the position of a 1-soliton. In particular, if the entries from $p - 1$ to $p + 3$ are* '11011', *then p is the position of a 1-soliton.*

As for the movement of the 1-soliton, we have

Proposition 3.6. *The position of a 1-soliton is updated by* $+1$ *or* -1.

Proof. First we assume that the distance between the 1-solitons is at least 3. Let p be the position of a 1-soliton at t. The entry at $p + 1$ must be '0'. If the entry at $p + 2$ is '1', then the entries from p to $p + 2$ form a pattern '010' at $t + 1$ and we see that the position changes by $+1$.

When the entry at $p+2$ is '0', then the entry at $p-1$ is '0' by proposition 3.5. The index of '0' at $p+2$ is either $-\infty$ or $-k$ ($k \geqslant 2$). If it is $-\infty$, the entries from p to $p+2$ become a '010' pattern at $t+1$ and we can regard the position as changing by +1. If it is $-k$, the index of '0' at $p-1$, $\ell_{p-1}(t)$, satisfies $-k+1 \leqslant \ell_{p-1}(t) \leqslant -1$, hence the entries from $p-2$ to $p+2$ are updated as either '01011' or '11011'. By proposition 3.5, the position of the soliton is $p-1$ and hence it changes by -1.

When the distance between the 1-solitons is just 2, they form a sequence such as $\ldots\underbrace{1010\cdots10}_{m\ \text{times}}\ldots$. Let p be the position of the leftmost 1-soliton. Then we repeat the same arguments as above and find that all the solitons move by +1 when the entry at $p+2m$ is '1' or '0' with index $-\infty$ and by -1 when it is '0' with index $-k$. □

4. Fundamental cycle of the pBBS

In this section, we consider the fundamental cycle (the shortest period) of a pBBS for a given initial state. A key idea to determine the fundamental cycle is to establish a relation between the pBBS and the one obtained from it by 10-elimination. For a given state of the pBBS, we obtain a pattern of the state of a smaller pBBS by 10-elimination. A soliton with length L ($L \geqslant 2$) turns into a soliton with length $L-1$, and a 1-soliton disappears. However, it is convenient to think of it as turning into a 0-*soliton*—soliton with length 0—which has no entry but has a *position*. To make the idea clear, consider the following example. A state

$$000111000\underline{1}000011\underline{1}011000000$$

contains two 1-solitons, the '1's which are underlined. By 10-elimination, we obtain a new pattern

$$000110000011100000.$$

In this new sequence, however, we can denote the places where 1-solitons existed before the 10-elimination as

$$0001100|00011|100000.$$

So we may think that 0-solitons exist between two consecutive entries separated by a '|'. Of course, the notion of a 0-*soliton* only has a meaning when we eliminate 1-solitons by 10-elimination. An advantage to considering the 0-solitons is that we can devise a reverse operation for 10-elimination, i.e., we can reconstruct the original state from the new sequence if we allow the existence of 0-solitons.

When there are solitons with the same length, we can define the *effective distance* between two of them with respect to the notion of a 0-soliton.

Definition 4.1. *When there are two solitons with the same length, we perform 10-elimination repeatedly and turn them into 0-solitons. We denote the distance between these two 0-solitons by the effective distance between them.*

Here the distance between the two 0-solitons is understood as the number of entries between them. For example, the effective distance between the above two 1-solitons is 5. An important property of the effective distance is

Proposition 4.1. *The effective distance between two solitons of the same type does not change in time.*

Proof. By lemma 2.2, 10-elimination commutes with the updating rule of a pBBS. Hence it is enough to prove the proposition for two 1-solitons.

If there is no soliton with length greater than 1, the proposition is trivial. If the 1-solitons themselves form blocks, i.e., if they do not belong to a block together with a larger soliton, the statement of the proposition is also obvious because they will move by +1 and the number of '10's between the two solitons does not change at the next time step.

At time t, let us consider a block to which one of the 1-solitons belongs. The block is composed of consecutive '1's and '0's and they change to '0's and '1's at the next time step. Let p_l and p_r, respectively, be the positions of '1' at the left edge of the block and of the '0' at the right edge. Suppose that there are m_r '10's on the right of the 1-soliton in the block, and m_l '10's on the left of it. (Hence there are $m_r + m_l + 1$ '10's in the block.) When we eliminate '10's at t the number of entries in the block decreases by $2(m_r + m_l + 1)$. Let the distance between the 0-soliton and the right (left) edge of the reduced block be L_r (L_l).

From proposition 3.6, the 1-soliton moves by +1 or −1 in the next time step. If it moves by +1, the number of '10's in the block which exist to the right of the 1-soliton do not change. (We take into account the '10' at the right edge of the block as well.) Since the total number of '10's does not change in a block at t, the number of '10's to the left of the 1-soliton does not change either. On the other hand, when it moves by −1, the number of '10's to its right increases by 1 and the number of those on its left decreases by 1. Hence, if we eliminate '10's at $t+1$, the distance between the 0-soliton and the '1' next to the right edge of the block, which was originally at the position $p_r - 1$, is also L_r in both cases. It is also clear that the distance between the 0-soliton and the '0' next to the left edge, which was at the position $p_l + 1$, is also L_l. Therefore, when we perform 10-elimination, the position of the 0-soliton in the reduced block does not change by the updating rule, where the reduced block at $t + 1$ is defined as the sequence starting from the '0' which was at $p_l + 1$ to the '1' which was at $p_r - 1$. Since the distance between the reduced blocks does not change under time evolution, even for a block of 0-soliton obtained from a block of 1-solitons, the effective distance between two 1-solitons does not change in time evolution, which completes the proof. □

From the above proposition, we know that the effective distance is also a conserved quantity of the pBBS.

The pBBS sometimes has some hidden symmetry which makes it difficult to determine the fundamental cycle. The symmetry appears when there are solitons with the same length. Suppose that there is more than one soliton with length L. By applying 10-elimination for L times, these solitons turn into 0-solitons. Let N_L be the size of the reduced pBBS. We denote the positions of the 0-solitons in this reduced pBBS by $p_1, p_2, \ldots, p_k \in \mathbb{Z}_{N_L}$. Here p_i is understood as the position of the right box adjacent to the ith 0-soliton. Let s_L be the smallest positive integer which satisfies

$$\{p_1, p_2, \ldots, p_k\} = \{p_1 + s_L, p_2 + s_L, \ldots, p_k + s_L\}.$$

(Note that the positions are defined modulo N_L.) The number s_L $(1 \leqslant s_L \leqslant N_L)$ is a divisor of N_L and we put $m_L := N_L/s_L \in \mathbb{Z}$. By proposition 4.1, m_L does not change in time. In this situation, we define the effective symmetry as follows.

Definition 4.2. *Let m_L be the positive integer defined above. Then, the solitons with length L in the pBBS are said to have effective translational symmetry of order m_L.*

For example, a sequence

1100001000111010011000010

has three 1-solitons. By 10-elimination, we have

1000|0011|0100|,

i.e.; $N_1 = 15, s_1 = 5$ and hence the 1-solitons have effective translational symmetry of order 3.

Now we fix the system and its initial state at $t = 0$. As in section 1, we consider a pBBS with N boxes. We assume that the initial state has n_j solitons with length L_j $(j = 1, 2, \ldots, s)$, where $L_1 > L_2 > \cdots > L_s$. We also put $\ell_0 := N - \sum_{j=1}^{l} 2p_j = N - \sum_{j=1}^{s} 2n_j L_j$ and

$$\ell_j := L_j - L_{j+1} \qquad (j = 1, 2, \ldots, s-1). \tag{4.1}$$

The initial state turns to a state of a smaller pBBS by 10-elimination. By kth *pBBS*, we denote the pBBS which has as an initial state the one obtained by L_{k+1} times 10-elimination performed on the original initial state. The position of the origin of the kth pBBS can be determined arbitrarily. The size of the kth pBBS, N_k, is given as

$$N_k := \ell_0 + 2n_1(L_1 - L_{k+1}) + 2n_2(L_2 - L_{k+1}) + \cdots + 2n_k(L_k - L_{k+1}). \tag{4.2}$$

In particular, $N_0 = \ell_0$.

The pBBS is then updated according to the time evolution rule and after some time period it will take the same pattern as the initial state except for some translation. We call such a period a *relative cycle*. The shortest relative cycle is called *the fundamental relative cycle*. Apparently, the fundamental cycle is a relative cycle and an integer multiple of the fundamental relative cycle.

The following lemma is used to establish the relation between the fundamental cycle of the kth pBBS and the fundamental relative cycle of the $(k + 1)$th pBBS.

Lemma 4.1. *When the smallest soliton has length $\ell \geqslant 2$ in a pBBS, the fundamental relative cycle of the pBBS coincides with that of the reduced pBBS obtained by 10-elimination.*

Proof. From lemma 2.2, we have that the fundamental relative cycle of the pBBS is the relative cycle of the reduced pBBS. Since the number of solitons does not change by 10-elimination in this case, we can reconstruct the pattern from the reduced pattern by adding '10's to the left of the solitons. Thus the fundamental relative cycle of the reduced pBBS is also a relative cycle of the original pBBS. □

For a pattern in a pBBS which contains at t a 1-soliton, we obtain a pattern for a reduced pBBS by 10-elimination. The 1-soliton turns into a 0-soliton in that pattern. If we now shift this pattern such that the position of this 0-soliton is situated at the left (the midpoint between the positions 0 and 1), we can define a map from a sequence of the pBBS to a sequence of the reduced pBBS. We shall denote this map by $\Psi(p(t))$. Here $p(t)$ denotes the position of the 1-soliton. For example, a sequence

$$000111000\underline{110}110000$$

is mapped by $\Psi(11)$ to

$$|100000011001$$

where '|' denotes the position of the 0-soliton.

An important property of this map is that it commutes with time evolution, namely,

Lemma 4.2. *Let $\mathcal{S}_t$ be a sequence of a pBBS at t with a 1-soliton at position $p(t)$. Then we have*

$$\Psi(p(t+1))\mathcal{S}_{t+1} = \hat{T}\,(\Psi(p(t))\mathcal{S}_t) \tag{4.3}$$

where $\hat{T}$ denotes the time evolution in the reduced pBBS.

Proof. By lemma 2.2, the sequence $\Psi(p(t+1))\mathcal{S}_{t+1}$ is equivalent to $\hat{T}\,(\Psi(p(t))\mathcal{S}_t)$. Hence it suffices to show that the position of one specific soliton coincides in both sequences.

The sequence $\mathcal{S}_t$ consists of blocks. First we consider the case where the 1-soliton at $p(t)$ itself constitutes a block. Let L_l be the distance between the 1-soliton at t and the right edge of the block nearest to it on the left. Since the position of the largest soliton in the block becomes the right edge of the block (corollary 3.1), after 10-elimination, there are $L_l - 1$ '0's between the 0-soliton and the nearest soliton on its left. Namely, the position of the soliton nearest to it on the left is $L_l - 2$. On the other hand, the blocks of the sequence $\Psi(p(t))\mathcal{S}_t$ are obtained by eliminating all the '10's, changing indices k to $k-1$ and $-k$ to $-k+1$ ($k \geqslant 2$), and letting the position of the 0-soliton be the midpoint between the entries at positions 0 and 1. Since the updating rule is equivalent to changing '0's in the block to '1's and '1's to '0', we find that the position of the largest soliton in the block becomes $L_l - 2$ at $t+1$. Thus, we find that relation (4.3) holds.

Next we consider the case where the 1-soliton belongs to a block together with a larger soliton. As shown in the proof of proposition 4.1, the distance (number of entries) between the 0-soliton and the right edge of the block at t coincides with that between the position of the largest soliton and the 0-soliton at $t+1$. Since the distance from the 0-soliton to the right edge is equal to the position of the largest soliton of the block in $\hat{T}\,(\Psi(p(t))\mathcal{S}_t)$, the position of the largest soliton in the block in $\Psi(p(t+1))\mathcal{S}_{t+1}$ coincides with that in $\hat{T}\,(\Psi(p(t))\mathcal{S}_t)$, which completes the proof. □

From this lemma, and because the effective distance between 1-solitons does not change, by proposition 4.1, we find that the positions of all the 0-solitons in $\Psi(p(t))\mathcal{S}_t$ coincide with those in $\Psi(p(t+1))\mathcal{S}_{t+1}$. We state this fact as a lemma.

Lemma 4.3. *The positions of all the 0-solitons in* $\Psi(p(t))\mathcal{S}_t$ *coincide with those in* $\Psi(p(t+1))\mathcal{S}_{t+1}$.

Using these lemmas, we can prove the key proposition.

Proposition 4.2. *The fundamental cycle of the kth pBBS is a relative cycle of the* $(k+1)$*th pBBS. If the smallest solitons in the* $(k+1)$*th pBBS do not have effective translational symmetry, it is the fundamental relative cycle of the* $(k+1)$*th pBBS.*

Proof. We denote the pattern at t of the $(k+1)$th pBBS by $\mathcal{S}_t$. We choose one of the 1-solitons and denote its position at t by $p(t)$. Without loss of generality, we can assume the sequence of the kth system at $t=0$ to coincide with $\Psi(p(0))\mathcal{S}_0$. By lemma 4.2, we have

$$\Psi(p(t+1))\mathcal{S}_{t+1} = \hat{T}(\Psi(p(t))\mathcal{S}_t)$$

where $\hat{T}$ is the time evolution operator for the kth pBBS. Therefore we find

$$\Psi(p(t))\mathcal{S}_t = \hat{T}^t\Psi(p(0))\mathcal{S}_0. \tag{4.4}$$

If T is the fundamental cycle of the kth pBBS, (4.4) implies

$$\Psi(p(T))\mathcal{S}_T = \Psi(p(0))\mathcal{S}_0.$$

Since by lemma 4.3 the above relation holds up to the position of 0-solitons, $\mathcal{S}_T$ is equivalent to $\mathcal{S}_0$. Therefore T is a relative cycle of the $(k+1)$th pBBS. Conversely, if T is the fundamental relative cycle of the $(k+1)$th pBBS, $\mathcal{S}_T$ is equivalent to $\mathcal{S}_0$, and ${}^{\exists}p$, $\Psi(p(T))\mathcal{S}_T = \Psi(p)\mathcal{S}_0$. However, if the 1-solitons in the $(k+1)$th pBBS do not have the effective translational symmetry, p must be $p(0)$, by lemma 4.3. Thus we find

$$\hat{T}^T\Psi(p(0))\mathcal{S}_0 = \Psi(p(0))\mathcal{S}_0$$

which means that T is a cycle of the kth pBBS. Since T is the shortest relative cycle of the $(k+1)$th pBBS and the fundamental cycle of the kth pBBS is a relative cycle of the $(k+1)$th pBBS, T is the fundamental cycle of the kth pBBS. □

Now we determine the fundamental cycle of the pBBS for a given initial state. We divide the initial states into two cases.

Case 1 (solitons without effective translational symmetry). First we assume that no soliton has effective translational symmetry. Since there are only solitons with length ℓ_1 and the size of the system is N_1, the fundamental cycle of the first system t_1 is obtained as

$$t_1 = \frac{\mathrm{LCM}(N_1, \ell_1)}{\ell_1} \tag{4.5}$$

where $\mathrm{LCM}(N_1, \ell_1)$ denotes the least common multiple of N_1 and ℓ_1. During this period, a soliton passes the origin (the position of the 0-soliton) of the first pBBS exactly r_1 times, where

$$\begin{aligned} r_1 &= \frac{t_1 \ell_1}{N_1} \\ &= \frac{\mathrm{LCM}(N_1, \ell_1)}{N_1}. \end{aligned} \tag{4.6}$$

From proposition 4.2, the fundamental relative cycle of the second pBBS is t_1. The fundamental cycle T_2 is an integer multiple of the relative cycle. To determine T_2, we look for the distance s_2 over which the largest soliton will move during the period t_1. Since the largest soliton in the second pBBS has length $\ell_1 + \ell_2$ and it collides r_1 times with each of the smaller solitons during the fundamental relative cycle, by corollary 3.1, we obtain

$$s_2 = t_1(\ell_1 + \ell_2) + 2r_1 n_2 \ell_2. \tag{4.7}$$

Hence, the fundamental cycle of the second system is given as $T_2 = t_1 t_2$, where

$$t_2 = \frac{\mathrm{LCM}(N_2, s_2)}{s_2}. \tag{4.8}$$

During the period T_2, the largest soliton passes r_2 times through the origin of the second pBBS, where

$$\begin{aligned} r_2 &= \frac{t_2 s_2}{N_2} \\ &= \frac{\mathrm{LCM}(N_2, s_2)}{N_2}. \end{aligned} \tag{4.9}$$

In a similar manner, the largest soliton in the third pBBS has length $\ell_1 + \ell_2 + \ell_3$ and its shift during the period T_2 is given as

$$s_3 = t_1 t_2(\ell_2 + \ell_2 + \ell_3) + 2r_1 n_2 t_2(\ell_2 + \ell_3) + 2r_2 n_3 \ell_3. \tag{4.10}$$

The fundamental cycle T_3 is given as $T_3 = t_1 t_2 t_3$, where

$$t_3 = \frac{\mathrm{LCM}(N_3, s_3)}{s_3} \tag{4.11}$$

and it passes the origin r_3 times, where

$$r_3 = \frac{\mathrm{LCM}(N_3, s_3)}{N_3}. \tag{4.12}$$

Repeating the above procedure, we find that the fundamental cycle of the pBBS is given by the following theorem

Theorem 4.1. *When there is no effective translational symmetry of solitons in a pBBS, the fundamental cycle T of the pBBS is given as*

$$T = \prod_{k=1}^{s} t_k \tag{4.13}$$

where t_k is obtained recursively as

$$s_1 = \ell_1 \tag{4.14}$$

$$t_1 = \frac{\mathrm{LCM}(N_1, s_1)}{s_1} \tag{4.15}$$

$$r_1 = \frac{\mathrm{LCM}(N_1, s_1)}{N_1} \tag{4.16}$$

$$\begin{aligned} s_k = {} & t_1 t_2 \cdots t_{k-1}(\ell_1 + \ell_2 + \cdots + \ell_k) + 2r_1 n_2 t_2 t_3 \cdots t_{k-1}(\ell_2 + \ell_3 + \cdots + \ell_k) \\ & + 2r_2 n_3 t_3 \cdots t_{k-1}(\ell_3 + \ell_4 + \cdots + \ell_k) + \cdots + 2r_{k-1} n_k \ell_k \end{aligned} \tag{4.17}$$

$$t_k = \frac{\mathrm{LCM}(N_k, s_k)}{s_k} \tag{4.18}$$

$$r_k = \frac{\mathrm{LCM}(N_k, s_k)}{N_k}. \tag{4.19}$$

Here ℓ_k and N_k $(k = 1, 2, \ldots, s)$ are defined as in (4.1) and (4.2).

In order to rewrite this formula in a compact form, we first note that

$$s_k = r_{k-1} N_k + \frac{t_1 t_2 \cdots t_{k-1} \ell_k \ell_0}{N_{k-1}}. \tag{4.20}$$

Equation (4.20) is proved by induction. We give a proof in appendix B. Noting the identity

$$\frac{\mathrm{LCM}(s + jN, N)}{s + jN} = \frac{\mathrm{LCM}(s, N)}{s} \qquad \forall j \in \mathbb{Z}$$

we have

$$t_k = \frac{\mathrm{LCM}(N_k, q_k)}{q_k} \tag{4.21}$$

where $q_1 = \ell_1$ and

$$q_k := \frac{t_1 t_2 \cdots t_{k-1} \ell_k \ell_0}{N_{k-1}} \qquad (k \geqslant 2). \tag{4.22}$$

The map $\mathrm{LCM}(\mathbb{Z}_+ \times \mathbb{Z}_+ \to \mathbb{Z}_+)$ is naturally extended to the map $\mathbb{Q}_+ \times \mathbb{Q}_+ \to \mathbb{Q}_+$ as

$$\begin{aligned} & \mathrm{LCM}(2^{a_1} 3^{a_2} 5^{a_3} 7^{a_4} \cdots, 2^{b_1} 3^{b_2} 5^{b_3} 7^{b_4} \cdots) \\ & \quad = 2^{\max[a_1, b_1]} 3^{\max[a_2, b_2]} 5^{\max[a_3, b_3]} 7^{\max[a_4, b_4]} \cdots \qquad (a_j, b_j \in \mathbb{Z}). \end{aligned} \tag{4.23}$$

We also define $\mathrm{LCM}(a, b, c) := \mathrm{LCM}(\mathrm{LCM}(a, b), c)$, etc as $\mathrm{LCM}(\mathrm{LCM}(a, b), c) = \mathrm{LCM}(a, \mathrm{LCM}(b, c))$. From (4.21) and (4.22), we obtain

$$t_k = \mathrm{LCM}\left(\frac{N_{k-1} N_k}{t_1 t_2 \cdots t_{k-1} \ell_k \ell_0}, 1\right)$$

and

$$\prod_{j=1}^{k} t_j = \mathrm{LCM}\left(\frac{N_{k-1} N_k}{\ell_k \ell_0}, t_1 t_2 \cdots t_{k-1}\right).$$

Thus we have

$$\prod_{j=1}^{s} t_j = \mathrm{LCM}\left(\frac{N_{s-1}N_s}{\ell_s\ell_0}, t_1t_2\cdots t_{s-1}\right)$$
$$= \mathrm{LCM}\left(\frac{N_{s-1}N_s}{\ell_s\ell_0}, \frac{N_{s-2}N_{s-1}}{\ell_{s-1}\ell_0}, t_1t_2\cdots t_{s-2}\right)$$
$$= \cdots.$$

Hence, from (4.13), we obtain

Corollary 4.1

$$T = \mathrm{LCM}\left(\frac{N_sN_{s-1}}{\ell_s\ell_0}, \frac{N_{s-1}N_{s-2}}{\ell_{s-1}\ell_0}, \ldots, \frac{N_1N_0}{\ell_1\ell_0}, 1\right). \tag{4.24}$$

For example, the pBBS with the initial state

0011101110010001111000110100000

has four types of solitons with lengths 5, 4, 2 and 1. The values of the parameters are calculated as $\ell_1 = 1$, $\ell_2 = 2$, $\ell_3 = 1$, $\ell_4 = 1$, $N_0 \equiv \ell_0 = 4$, $N_1 = 6$, $N_2 = 14$, $N_3 = 20$ and $N_4 = 32$. Hence its fundamental cycle is

$$T = \mathrm{LCM}\left(\frac{32\times 20}{1\times 4}, \frac{20\times 14}{1\times 4}, \frac{14\times 6}{2\times 4}, \frac{6\times 4}{1\times 4}, 1\right)$$
$$= \mathrm{LCM}\left(160, 70, \frac{21}{2}, 6, 1\right)$$
$$= 2^5\times 3\times 5\times 7$$
$$= 3360.$$

Case 2 (solitons with effective translational symmetry). The main difference with the previous case is that the fundamental cycle of the kth pBBS is a relative cycle of the $(k+1)$th pBBS, but not necessarily the fundamental relative cycle. When the largest solitons do not have effective symmetry, however, the fundamental relative cycle can be determined easily and the above arguments require only slight modification. In the first pBBS the 0-solitons, obtained by 10-elimination on the second pBBS, are arranged periodically. If we put $N_1^* := \frac{N_1}{m_2}$, N_1^* is the period of the translational symmetry. Hence the fundamental relative cycle of the second pBBS t_1^* is given as

$$t_1^* = \frac{\mathrm{LCM}(N_1^*, \ell_1)}{\ell_1}. \tag{4.25}$$

If we put

$$r_1^* = \frac{\mathrm{LCM}(N_1^*, \ell_1)}{N_1^*} \tag{4.26}$$

the largest soliton collides with 0-solitons exactly $\frac{r_1^* n_2}{m_2}$ times during the period t_1^*. Hence, in the second pBBS, the largest soliton moves the distance

$$s_2^* = t_1^*(\ell_1+\ell_2) + 2r_1^* n_2^* \ell_2 \tag{4.27}$$

where $n_2^* = \frac{n_2}{m_2}$. Similarly, the fundamental relative cycle of the third pBBS is

$$t_2^* = \frac{\mathrm{LCM}(N_2^*, s_2^*)}{s_2^*}. \tag{4.28}$$

We can proceed in a similar manner as above and find that all the formulae given in theorem 4.1 hold with the change $N_j \to N_j^*, t_j \to t_j^*$ etc, that is, the fundamental cycle T of the pBBS is given as

$$T = \prod_{k=1}^{s} t_k^* \tag{4.29}$$

where t_k^* is obtained recursively as

$$s_1^* = \ell_1 \tag{4.30}$$

$$t_1^* = \frac{\mathrm{LCM}(N_1^*, s_1^*)}{s_1^*} \tag{4.31}$$

$$r_1^* = \frac{\mathrm{LCM}(N_1^*, s_1^*)}{N_1^*} \tag{4.32}$$

$$s_k^* = t_1^* t_2^* \cdots t_{k-1}^* (\ell_1 + \ell_2 + \cdots + \ell_k) + 2r_1^* n_2^* t_2^* t_3^* \cdots t_{k-1}^* (\ell_2 + \ell_3 + \cdots + \ell_k) \\ + 2r_2^* n_3^* t_3^* \cdots t_{k-1}^* (\ell_3 + \ell_4 + \cdots + \ell_k) + \cdots + 2r_{k-1}^* n_k^* \ell_k \tag{4.33}$$

$$r_k^* = \frac{\mathrm{LCM}(N_k^*, s_k^*)}{N_k^*} \tag{4.34}$$

$$t_k^* = \frac{\mathrm{LCM}(N_k^*, s_k^*)}{s_k^*}. \tag{4.35}$$

Here $N_k^* = \frac{N_k}{m_{k+1}}$, $n_k^* = \frac{n_k}{m_k}$, m_k is the order of the effective translational symmetry of the kth largest solitons for $1 \leqslant k \leqslant s$, $m_{s+1} = 1$, and ℓ_k and N_k $(k = 1, 2, \ldots, s)$ are defined as in (4.1) and (4.2).

However, as in this case, there is no relation like (4.20), so far we have not found a compact expression like (4.24).

When the largest soliton has effective translational symmetry, the determination of the fundamental cycle becomes a little complicated. For example, the sequence of the first pBBS

|1100|0011|0000 at t

is updated as

|0011|0000|1100 at $t + 1$

where '|' denotes the 0-solitons obtained from the second pBBS. Apparently, the fundamental relative cycle of the second pBBS is just one time step. Although the largest soliton moves only two boxes, the distance between a pattern at t and the same pattern at $t + 1$ is 8. Hence, in this example, we have to think of the largest soliton as being shifted by a distance of eight boxes. In general, when we shift the largest solitons by $N_1^* = \frac{N_1}{\mathrm{LCM}(m_1, m_2)}$, the sequences coincide up to translation. Here m_k denotes the order of the effective translational symmetry of the kth largest solitons. Then the fundamental relative cycle of the second pBBS is given as

$$t_1^* = \frac{\mathrm{LCM}(N_1^*, \ell_1)}{\ell_1}. \tag{4.36}$$

The distance separating it from the same pattern, d, is calculated as follows. Let $(p_1, q_1) \in \mathbb{Z} \times \mathbb{Z}$ be a solution to the Diophantine equation

$$t_1 \ell_1 + \frac{N_1}{m_1} p_1 = \frac{N_1}{m_2} q_1. \tag{4.37}$$

Then

$$d = \frac{N_1}{m_2} q_1 \quad \left(= t_1 \ell_1 + \frac{N_1}{m_1} p_1 \right) \qquad \text{modulo } N_1.$$

In this expression, q_1 is the number of 0-solitons between the largest soliton at t and the 'corresponding' largest soliton at $t + t_1^*$; $p_1 n_1^*$ is the number of largest solitons between the largest soliton at $t + t_1^*$ which 'really' moved during the time period t_1^* and the corresponding largest soliton at $t + t_1^*$. Hence, in the second pBBS, the shift of the largest soliton is considered to be

$$s_2^* = t_1^*(\ell_1 + \ell_2) + 2(p_1 n_1^* + q_1 n_2^*)\ell_2 + p_1 \frac{N_1}{m_1}. \tag{4.38}$$

When $k_2 := \mathrm{GCD}(m_1, m_2) \neq 1$, the second pBBS also has translational symmetry and the fundamental relative cycle of the third pBBS becomes $t_1^* t_2^*$, where

$$t_2^* = \frac{\mathrm{LCM}(N_2^*, s_2^*)}{s_2^*} \tag{4.39}$$

$$N_2^* = \frac{N_2}{\mathrm{LCM}(k_2, m_3)}. \tag{4.40}$$

The shift of the largest solitons in the third pBBS is obtained in a similar manner as

$$\begin{aligned} s_3^* = t_1^* t_2^* (\ell_1 + \ell_2 + \ell_3) + t_2^* \left(2(p_1 n_1^* + q_1 n_2^*)(\ell_2 + \ell_3) + p_1 \frac{N_1}{m_1} \right) \\ + 2\left(\frac{n_1 + n_2}{k_2} p_2 + n_3^* q_2 \right) \ell_3 + \frac{N_2}{k_2} p_2 \end{aligned} \tag{4.41}$$

where $(p_2, q_2) \in \mathbb{Z} \times \mathbb{Z}$ is a solution to the Diophantine equation

$$t_2^* s_2^* + \frac{N_2}{k_2} p_2 = \frac{N_2}{m_3} q_2. \tag{4.42}$$

The fundamental relative cycle of the fourth pBBS is obtained from s_3^* in a similar manner to (4.39). Repeating the same procedure, we obtain the fundamental cycle of the pBBS as $T = \prod_{j=1}^{s} t_j^*$, where

$$t_j^* = \frac{\mathrm{LCM}(N_j^*, s_j^*)}{s_j^*} \tag{4.43}$$

$$N_j^* = \frac{N_j}{\mathrm{LCM}(k_j, m_{j+1})} \qquad (m_{s+1} := 1) \tag{4.44}$$

$$k_j = \mathrm{GCD}(m_1, m_2, \ldots, m_j) \tag{4.45}$$

$$s_1^* = \ell_1 \tag{4.46}$$

$$\begin{aligned} s_j^* = \prod_{i=1}^{j-1} t_i^* \left(\sum_{i=1}^{j} \ell_j \right) + \prod_{i=2}^{j-1} t_i^* \left[2(n_1^* p_1 + n_2^* q_1) \left(\sum_{i=2}^{j} \ell_j \right) + \frac{N_1}{m_1} p_1 \right] \\ + \prod_{i=3}^{j-1} t_i^* \left[2\left(\frac{n_1 + n_2}{k_2} p_2 + n_3^* q_2 \right) \left(\sum_{i=3}^{j} \ell_j \right) + \frac{N_2}{k_2} \right] \\ + 2\left(\frac{\sum_{i=1}^{j-1} n_i}{k_{j-1}} p_{j-1} + n_j^* q_{j-1} \right) \ell_j + \frac{N_{j-1}}{k_{j-1}} p_{j-1} \end{aligned} \tag{4.47}$$

and $(p_j, q_j) \in \mathbb{Z} \times \mathbb{Z}$ is a solution to the Diophantine equation

$$t_j^* s_j^* + \frac{N_j}{k_j} p_j = \frac{N_j}{m_{j+1}} q_j. \tag{4.48}$$

Therefore we have found a formula to obtain the fundamental cycle of the pBBS for an arbitrary initial state.

Theorem 4.2. *The fundamental cycle T of the pBBS is given as*

$$T = \prod_{j=1}^{s} t_j^* \tag{4.49}$$

where t_j^* is obtained recursively from (4.35) if the largest solitons have no effective translational symmetry, or from (4.43) in the general case.

5. Concluding remarks

In this paper, we have shown a formula to determine the fundamental cycle of a pBBS for a given initial state. The pBBS is obtained from the ultradiscretization of the discrete Toda equation with periodic boundary condition [10]. Since the Toda equation with periodic boundary condition has quasiperiodic solutions which are expressed by theta functions associated with hyperelliptic curves [11, 12], one may expect that the fundamental cycle has some relation with the period matrices of the theta functions. Establishing such a relation is a problem we wish to address in the future.

Although we have treated the pBBS with box capacity 1 and with only one kind of ball, it can be extended to the case with box capacity larger than 1 and many kinds of balls. The determination of the fundamental cycle for such an extended pBBS is also a future problem.

Finally, we wish to point out one important property of the original BBS which can be derived from the results in the previous sections. Note that the BBS corresponds to the case of the pBBS where the number of boxes $N \to +\infty$. The solitons are defined in exactly the same way and they are also conserved in time. From corollary 3.1, lemmas 4.2 and 4.3, we obtain

Proposition 5.1. *In the original BBS, after sufficient time steps, the solitons are arranged according to the order of their lengths and move freely.*

In the BBS, solitons defined here behave as *solitons* in nonlinear differential equations, which justifies the use of the word 'soliton'.

Acknowledgments

The authors are grateful to Professor J Satsuma and Dr R Willox for helpful comments on the present work.

Appendix A. Proof of proposition 2.2

We use the terminology of sections 2 and 3. For a given state of the pBBS, we construct a state of the kth pBBS by performing 10-elimination L_{k+1} times. In particular, after L_1 eliminations, we obtain a sequence of ℓ_0 consecutive '0's. We then fix the position of one such '0'. To obtain a state of the first system, we should put n_1 0-solitons next to these consecutive '0's. The number of such possible configurations is $\binom{\ell_0+n_1-1}{n_1}$. Each configuration corresponds to a state of the first system. To obtain a state of the second system, we put n_2 0-solitons next to the previous state. The number of possible configurations is $\binom{N_1+n_2-1}{n_2}$. Repeating this procedure, we see that the number of states for a fixed '0' is

$$\binom{\ell_0+n_1-1}{n_1}\binom{N_1+n_2-1}{n_2}\binom{N_2+n_3-1}{n_3}\cdots\binom{N_{s-1}+n_s-1}{n_s}.$$

There are N possibilities for the position of the fixed '0', but any one of ℓ_0 '0's gives the same results. Thus we find that the number of states $\Omega(N;\{L_j,n_j\})$ is given as

$$\Omega(N;\{L_j,n_j\}) = \frac{N}{\ell_0}\prod_{i=1}^{s}\binom{N_{i-1}+n_i-1}{n_i}. \qquad \square$$

Appendix B. Proof of (4.20)

We prove (4.20) by induction. Since $N_1 = \ell_0 + 2n_1\ell_1$, $N_2 = \ell_0 + 2n_1\ell_1 + 2(n_1+n_2)\ell_2$ and $r_1N_1 = t_1\ell_1$, we have

$$\begin{aligned} s_2 &= t_1\left[(\ell_1+\ell_2) + \frac{r_1}{t_1}2n_2\ell_2\right] \\ &= t_1\left[(\ell_1+\ell_2) + \frac{\ell_1}{N_1}2n_2\ell_2\right] \\ &= \frac{t_1}{N_1}\left[\ell_1(\ell_0+2n_1(\ell_1+\ell_2)+2n_2\ell_2)+\ell_0\ell_2\right] \\ &= r_1N_2 + \frac{t_1\ell_2\ell_0}{N_1}. \end{aligned}$$

Hence (4.20) is true for $k = 2$. Suppose that (4.20) holds up to $k = m-1$ ($m \geqslant 3$). We rewrite s_m as

$$\begin{aligned} s_m &= t_1t_2\cdots t_{m-1}(\ell_1+\ell_2+\cdots+\ell_m) + 2r_1t_2t_3\cdots t_{m-1}(\ell_2+\cdots+\ell_m)+\cdots+2r_{m-1}n_m\ell_m \\ &= t_{m-1}[s_{m-1} + ((t_1t_2\cdots t_{m-2}) + (2r_1t_2\cdots t_{m-2})n_2 \\ &\quad + \cdots + 2r_{m-2}n_{m-1})\ell_m] + 2r_{m-1}n_m\ell_m. \end{aligned} \tag{B.1}$$

Using the relation $r_kN_k = t_ks_k$ ($\forall k$) and $N_m = N_{m-1} + \sum_{j=1}^{m} 2n_j\ell_m$, we have

$$\begin{aligned} s_m = r_{m-1}N_m + t_{m-1}\ell_m \Bigg[& t_1t_2\cdots t_{m-2} + 2r_1t_2\cdots t_{m-2}n_2 \\ & + \cdots + 2r_{m-2}n_{m-1} - \frac{2s_{m-1}}{N_{m-1}}\left(\sum_{j=1}^{m-1}n_j\right)\Bigg]. \end{aligned} \tag{B.2}$$

By the induction hypothesis

$$s_{m-1} = r_{m-2}N_{m-1} + \frac{t_1t_2\cdots t_{m-2}\ell_{m-1}\ell_0}{N_{m-2}}$$

and

$$\begin{aligned} & 2r_{m-2}n_{m-1} - \frac{2s_{m-1}}{N_{m-1}}\left(\sum_{j=1}^{m-1}n_j\right) \\ &\quad = -\frac{2t_{m-1}s_{m-2}}{N_{m-2}}\left(\sum_{j=1}^{m-2}n_j\right) - t_1t_2\cdots t_{m-2}\left(\sum_{j=1}^{m-1}n_j\right)\frac{\ell_{m-1}\ell_0}{N_{m-1}N_{m-2}}. \end{aligned}$$

Consecutive use of the induction hypothesis yields

$$t_1 t_2 \cdots t_{m-2} + 2r_1 t_2 \cdots t_{m-2} n_2 + \cdots + 2r_{m-2} n_{m-1} - \frac{2s_{m-1}}{N_{m-1}} \left(\sum_{j=1}^{m-1} n_j \right)$$

$$= \left(\prod_{j=1}^{m-2} t_j \right) \ell_0 \left(\frac{1}{N_1} - \frac{2\ell_2 (n_1 + n_2)}{N_1} N_2 - \cdots - \frac{2\ell_{m-1} \left(\sum_{j=1}^{m-1} n_j \right)}{N_{m-2} N_{m-1}} \right)$$

$$= \left(\prod_{j=1}^{m-2} t_j \right) \ell_0 \left(\frac{1}{N_2} - \frac{2\ell_3 \left(\sum_{j=1}^{3} n_j \right)}{N_2 N_3} - \cdots - \frac{2\ell_{m-1} \left(\sum_{j=1}^{m-1} n_j \right)}{N_{m-2} N_{m-1}} \right)$$

$$= \cdots$$

$$= \frac{t_1 t_2 \cdots t_{m-2} \ell_0}{N_{m-1}}. \tag{B.3}$$

Substituting (B.3) into (B.2), we obtain

$$s_m = r_{m-1} N_m + \frac{t_1 t_2 \cdots t_{m-1} \ell_m \ell_0}{N_{m-1}}. \tag{B.4}$$

Therefore (4.20) is proved. □

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